

# Mechanisms for Multi-Scale Structures in Dense Degenerate Astrophysical Plasmas

N.L. Shatashvili<sup>1,2</sup> • S.M. Mahajan<sup>3</sup> •  
V.I. Berezhiani<sup>1,4</sup>

**Abstract** Two distinct routes lead to the creation of multi-scale equilibrium structures in dense degenerate plasmas, often met in astrophysical conditions. By analyzing an e-p-i plasma consisting of degenerate electrons and positrons with a small contamination of mobile classical ions, we show the creation of a new macro scale  $L_{\text{macro}}$  (controlled by ion concentration). The temperature and degeneracy enhancement effective inertia of bulk e-p components also makes the effective skin depths larger (much larger) than the standard skin depth. The emergence of these intermediate and macro scales lends immense richness to the process of structure formation, and vastly increases the channels for energy transformations. The possible role played by this mechanism in explaining the existence of large-scale structures in astrophysical objects with degenerate plasmas, is examined.

**Keywords** sun: evolution; stars: white dwarfs; plasmas

## 1 Introduction

Dense compressed plasmas, found in astrophysical and cosmological environments as well as in lab-

oratory experiments investigating interaction of intense lasers with high density plasma, are currently of great interest. In dense astrophysical objects like white and brown dwarfs, neutron stars, magnetars, and cores of giant planets, extreme conditions lead to high density degenerate matter (Sturrock 1971; Shapiro & Teukolsky 1973; Chandrasekhar 1931) ; (Chandrasekhar 1939; Ruderman & Sutherland 1975; Michel 1991; Michel 1982; Koester & Chanmugam 1990; Beloborodov & Thompson 2007). Most astrophysical plasmas usually contain ions in addition to degenerate electrons and positrons. Although there is no concrete evidence, the magnetospheres of rotating neutron stars are believed to contain electron-positron plasmas produced in the cusp regions of the stars due to intense electromagnetic radiation. Since protons or other ions may exist in such environments, three-component electron-positron-ion (e-p-i) plasmas can exist in pulsar magnetospheres (Begelman et al. 1984; Holcomb & Tajima 1989), (Berezhiani & Mahajan 1994,1995).

For typical dense plasmas, composed of ions, electrons, positrons, and/or holes (in the context of semiconductors), the lighter species (electrons, positrons, holes) are degenerate while the more massive ions, often, remain nondegenerate (classical).

It has been known that the nonlinear phenomena in e-p plasmas develop differently from their counterparts in the usual electron-ion system. The positron component could have a variety of origins: 1) positrons can be created in the interstellar medium due to the interaction of atoms and cosmic ray nuclei, 2) they can be introduced in a Tokamak e-i plasma by injecting bursts of neutral positronium atoms ( $e^+e^-$ ), which are then ionized by plasma; the annihilation time of positron in the plasma is long compared to typical particle confinement time (Uddin et al. 2015). The annihilation, which takes place in the interaction of matter (elec-

---

N.L. Shatashvili

<sup>1</sup>Andronikashvili Institute of Physics, TSU, Tbilisi 0177, Georgia

<sup>2</sup>Department of Physics, Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University (TSU), Tbilisi 0179, Georgia

S.M. Mahajan

<sup>3</sup>Institute for Fusion Studies, The University of Texas at Austin, Austin, Tx 78712

V.I. Berezhiani

<sup>1</sup>Andronikashvili Institute of Physics, TSU, Tbilisi 0177, Georgia

<sup>4</sup>School of Physics, Free University of Tbilisi, Georgia

trons) and anti-matter (positrons), usually occurs at much longer characteristic time scales compared with the time in which the collective interaction between the charged particles takes place (Berezhiani et al. 2015b, and references therein).

In the extremely low temperature three component e-p-i plasmas, studied in the context of pulsar magnetospheres in (Lominadze et al. 1986; Rizatto 1988) ; (Lakhina & Buti 1991; Halder et al. 2012), the de Broglie wavelength of the charge carriers can be comparable to the dimension of the systems. Such ultracold e-p-i plasma behaves like a Fermi gas and quantum mechanical effects are expected to play a significant role in their linear and nonlinear dynamics.

Pursuing the consequences of degeneracy in a multi-component plasma, it will be interesting to explore multiple-scale behavior accessible to such systems. An obvious goal will be to investigate if degeneracy can affect, for example, the dynamics of the star collapse when its multi-species atmosphere begins to contract. Analysis of multi-scale behavior can also be a guide in predicting various phenomena in the pre-compact era, or during the life of compact objects.

The principal determinant of degeneracy, the density, varies over many orders of magnitude in astro/cosmic settings. The rest frame e-p density near the pulsar surface is believed to be  $n \geq 10^{11} \text{ cm}^{-3}$  (Gedalin et al. 1998), while in the MeV epoch of the early Universe, it can be as high as  $n = 10^{32} \text{ cm}^{-3}$  (Weinberg 1972). Intense e-p pair creation takes place during the process of gravitational collapse of massive stars (Tsintsadze et al. 2003). It is argued that the gravitational collapse of the massive stars may lead to charge separation with the field strength exceeding the Schwinger limit resulting in e-p pair plasma creation with estimated density to be  $n = 10^{34} \text{ cm}^{-3}$  (Han et al. 2012). Superdense e-p plasma may also exist in GRB sources [ $n = (10^{30} - 10^{37}) \text{ cm}^{-3}$ ] (Aksenov et al. 2010).

Dense electron-positron plasma can be produced in laboratory conditions as well. Indeed, the modern petawatt lasers systems are already capable of producing ultrashort pulses with focal intensities  $I = 2 \times 10^{22} \text{ W/cm}^2$  (Yanovski et al. 2008). Pulses of even higher intensities exceeding  $I = 10^{26} \text{ W/cm}^2$  are likely to be available soon in lab or in the Lorentz boosted frames (Dunne 2006; Mourou et al. 2006) ; (Tajima 2014). Interaction of such pulses with gaseous or solid targets could lead to the generation of optically thin e-p plasmas ( $n \sim 10^{23} - 10^{28} \text{ cm}^{-3}$ ) denser than solid state systems (Wang et al. 2014).

In the highly compressed degenerate Fermi state (average inter-particle distance smaller than the thermal de Broglie wavelength), mutual interactions of the

plasma particles become unimportant, and plasma becomes more ideal as the density increases (Landau & Lifshitz 1980). A plasma may be considered cold (functionally zero temperature) if the thermal energy of the particles is much lower than the Fermi energy no matter how high the temperature really is (Russo 1988; Cercignani & Kremer 2002). The Fermi energy of degenerate electrons (positrons) is  $\epsilon_F = m_e c^2 \left[ (1 + R^2)^{1/2} - 1 \right]$ , where  $R = p_F / m_e c$  and  $p_F$  is the Fermi momentum determined by the rest-frame density  $p_F = m_e c (n/n_c)^{1/3}$ . Densities are normalized to the critical number-density  $n_c = 5.9 \times 10^{29} \text{ cm}^{-3}$  (Akbari-Moghanjoughi 2013).

The fluid models are frequently applied to study the large scale dynamics of relativistic multi-species plasmas (Gedalin 1996; Hazeltine & Mahajan 2002). Among such investigations the studies on *relaxed (equilibrium) states* have attracted considerable attention (Oliveira & Tajima 1995). Constrained minimization of fluid energy with appropriate helicity invariants has provided a variety of extremely interesting equilibrium configurations that have been exploited and found useful for understanding laboratory as well as astrophysical plasma systems [see e.g. (Oliveira & Tajima 1995; Woltjer 1958; Taylor 1974,1986; Sudan 1979), (Bhattacharjee & Dewar 1982; Dennis et al. 2014) and references therein]. Two particularly simple manifestations of this class of equilibria – *Beltrami states* – are: 1) The single Beltrami state,  $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ , discussed by Woltjer and Taylor (Woltjer 1958; Taylor 1974,1986) in the context of force free single fluid magnetohydrodynamics (MHD), and 2) a more general Double Beltrami State accessible to Hall MHD – a “two-fluid” system of ions and inertia-less electrons (Steinhauer & Ishida 1997); the latter has been investigated, in depth, by Mahajan and co-workers (Mahajan & Yoshida 1998; Mahajan et al. 2001), (Ohsaki et al. 2001,2002; Shukla & Mahajan 2004a;b), (Mahajan et al. 2002; Mahajan et al. 2005,2006), (Mahajan & Krishan 2005; Yoshida 2011). The content of the Beltrami conditions (derived by constrained minimization) is the alignment of a general “flow” with its vorticity. This, in turn, forces the total energy density of the system to distribute homogeneously (so called *Bernoulli condition*). The combined Beltrami-Bernoulli conditions define an equilibrium state that fits the notion of *relaxed state*, and constitutes a nontrivial helicity-bearing state. The characteristic number of a state is determined by the number of independent single Beltrami systems needed to construct it. For adequate description these states were named *Beltrami-Bernoulli States* (BB) (Mahajan & Yoshida 1998; Mahajan et al. 2001;

Berezhiani et al. 2015a); the latter ref. was also the first to study the effects of degeneracy (requiring Fermi-Dirac statistics) on BB states.

This paper, though worked out on the lines of (Berezhiani et al. 2015a), registers a major departure leading to the most important result of this article – by studying the BB states in an e-p-i (small dynamic ion contamination added to a primarily e-p plasma), we will demonstrate the creation of a new macroscopic length scale  $L_{\text{macro}}$  lying between the system size and the relatively small intrinsic scales (measured by the skin depths) of the system.

The new BB equilibrium is defined by: two relativistic Beltrami conditions (one for each dynamic degenerate species), one non-relativistic Beltrami condition for ion fluid, an appropriate Bernoulli condition, and Ampere’s law to close the set. This set of equations will lead to what may be called a *quadruple Beltrami system* [for multi-Beltrami systems see (Mahajan & Lingam 2015)]. The ions, though a small mobile component, play an essential role, they create an asymmetry in the electron-positron dynamics (to maintain charge neutrality, there is a larger concentration of electrons than positrons) and that asymmetry introduces a new and very important dynamical scale. This scale, though present in a classical non-degenerate plasma, turns out to be degeneracy dependent and could be vastly different from its classical counterpart. The significance of this scale in understanding the physics of relevant systems will be explored. Presence of mobile ions leads to “effective mass” asymmetry in electron and positron fluids, which, coupled with degeneracy-induced inertia, manifests in the existence of Quadruple Beltrami fields. Illustrative examples and application for concrete astrophysical systems will be suggested.

## 2 Model Equations

Charge neutrality in an e-p-i plasma of degenerate electrons (–), positrons (+) and a small mobile ion component, forces the following density relationships

$$N_0^- = N_0^+ + N_{i0} \quad \Rightarrow \quad \frac{N_0^+}{N_0^-} = 1 - \alpha \quad (1)$$

$$\text{with} \quad \alpha = \frac{N_{i0}}{N_0^-} ,$$

where  $\alpha$  labels the excess electron content. In this paper we will study only the  $\alpha \ll 1$  limit that we believe is the most pertinent for astro/cosmic settings.

The equation for Ion dynamics is standard (Berezhiani et al. 2015a). The  $e(p)$  dynamics will be

described by the relativistic degenerate fluid equations (Berezhiani et al. 2015a; Berezhiani et al. 2015b, and references therein): the continuity

$$\frac{\partial N^\pm}{\partial t} + \nabla \cdot (N^\pm \mathbf{V}_\pm) = 0 , \quad (2)$$

and the equation of motion

$$\frac{\partial}{\partial t} (G^\pm \mathbf{p}_\pm) + m_\pm c^2 \nabla (G^\pm \gamma_\pm) = q_\pm \mathbf{E} + \mathbf{V}_\pm \times \boldsymbol{\Omega}_\pm \quad (3)$$

where  $\mathbf{p}_\pm = \gamma_\pm m_\pm \mathbf{V}_\pm$  is the hydrodynamic momentum,  $n^\pm = N^\pm / \gamma_\pm$  is the rest-frame particle density ( $N^\pm$  denotes laboratory frame number density),  $q_\pm (m_\pm)$  is the charge (mass) of the positron (electron) fluid element,  $\mathbf{V}_\pm$  is the fluid velocity, and  $\gamma_\pm = (1 - V_\pm^2/c^2)^{-1/2}$ . Notice that the degeneracy effects manifest through the “effective mass” factor  $G^\pm = w_\pm / n^\pm m_\pm c^2$ , where  $w_\pm$  is an enthalpy per unit volume. The general expression for enthalpy  $w_\pm$  for arbitrary density and temperature (for a plasma described by local Dirac-Jüttner equilibrium distribution function) can be found in (Cercignani & Kremer 2002). For a fully degenerate (strongly degenerate)  $e(p)$  plasma, however, this very tedious expression smoothly transfers to the one with just density dependence, i.e.  $w_\pm \equiv w_\pm(n)$  (Berezhiani et al. 2015a). In fact  $w_\pm / n^\pm m_e c^2 = (1 + (R^\pm)^2)^{1/2}$ , where  $R^\pm [= p_{F\pm} / m_\pm c$  with  $p_{F\pm}$  being the Fermi momentum] have been defined earlier. The mass factor, then, is simply determined by the plasma rest frame density,  $G^\pm = [1 + (n^\pm / n_c)^{2/3}]^{1/2}$  for arbitrary  $n_\pm / n_c$ .

On taking the *curl* of these equations, one can cast them into an ideal vortex dynamics (Mahajan 2003 and references therein)

$$\frac{\partial}{\partial t} \boldsymbol{\Omega}_\pm = \nabla \times (\mathbf{V}_\pm \times \boldsymbol{\Omega}_\pm) , \quad (4)$$

in terms of the generalized (canonical) vorticities  $\boldsymbol{\Omega}_\pm = (q_\pm / c) \mathbf{B} + \nabla \times (G^\pm \mathbf{p}_\pm)$ .

It is, perhaps, the right juncture to re-emphasize that the so called plasma approximation for a degenerate  $e(p)$  assembly is valid if their average kinetic energy ( $\sim \epsilon_F^\pm$ ) is larger than the interaction energy ( $\sim e^2 (n_0^\pm)^{1/3}$ ). This condition is fulfilled for a sufficiently dense fluid when  $n_0^\pm \gg (2m_- e^2 / (3\pi^2)^{2/3} \hbar^2)^3 = 6.3 \cdot 10^{22} \text{ cm}^{-3}$ ; such a condition would imply  $R^\pm \gg 4.76 \cdot 10^{-3}$  (Berezhiani et al. 2015b).

The low frequency dynamics is, now, closed with Ampere's law

$$\nabla \times \mathbf{B} = \frac{4\pi e}{c} [(1 - \alpha) N^+ \mathbf{V}_+ - N^- \mathbf{V}_-] + \alpha N_i \mathbf{V}_i, \quad (5)$$

another relation between  $\mathbf{V}_i$ ,  $\mathbf{V}_\pm$  and  $\mathbf{B}$ . Notice that the small static/mobile ion population, represented by  $\alpha$  and  $\mathbf{V}_i$ , creates an asymmetry between the currents contributed by the electrons and positrons. This will be the source of a new scale-length that turns out to be much larger than the intrinsic electron and positron scale lengths (skin depths).

In this paper, we explore the combined effects of asymmetry and degeneracy on a special class of e-p-i equilibria known as the Beltrami-Bernoulli (BB) states. We expect to find large-scale structures originating in the ion-induced asymmetry. The e-p-i plasma system is, in some sense, more advanced and complete than the electron-ion system studied in a recent paper where it was shown that the electron degeneracy transformed BB states may be pertinent to advance our understanding of the evolution of certain astrophysical objects (Berezhiani et al. 2015a).

### 3 Equilibrium States in relativistic degenerate e-p-i Plasma

Before we write down the equations for the BB states, it is useful to express all physical quantities in normalized dimensionless form. In this paper, the density is normalized to  $N_0^-$  (the corresponding rest-frame density is  $n_0^-$ ); the magnetic field is normalized to some ambient measure  $|\mathbf{B}_0|$ ; all velocities are measured in terms of the corresponding Alfvén speed  $V_A = V_A^- = B_0 / \sqrt{8\pi n_0^- m^- G_0^-}$ ; all lengths [times] are normalized to the "effective" electron skin depth  $\lambda_{\text{eff}} [\lambda_{\text{eff}}/V_A]$ , where

$$\lambda_{\text{eff}} \equiv \lambda_{\text{eff}}^- = \frac{1}{\sqrt{2}} \frac{c}{\omega_p} = c \sqrt{\frac{m^- G_0^-}{8\pi n_0^- e^2}}; \quad (6)$$

$$G_0^\pm (n_0^\pm) = [1 + (R_0^\pm)^2]^{1/2} \quad \text{with} \quad (7)$$

$$R_0^\pm = \left( \frac{n_0^\pm}{n_c} \right)^{1/3}; \quad R_0^+ = (1 - \alpha)^{1/3} R_0^-. \quad (8)$$

The intrinsic skin depths, the natural length scales of the dynamics, are generally much shorter compared to

the system size. For the degenerate electron fluid, the effective mass goes from  $G_0^- (n_0^-) = 1 + \frac{1}{2} \left( \frac{n_0^-}{n_c} \right)^{2/3}$  in the non-relativistic limit ( $R_0^- \ll 1$ ) to  $G_0^- (n_0^-) = \left( \frac{n_0^-}{n_c} \right)^{1/3}$  in the ultra-relativistic regime ( $R_0^- \gg 1$ ).

By following the methodology of (Pino et al. 2010), where the BB states were derived for classical relativistic non-degenerate multi-fluid plasmas, we obtain the required set of equilibrium equations for the degenerate system (the primary difference is in the physics of  $G_\pm$  (Berezhiani et al. 2015a)): The Beltrami conditions

$$\mathbf{B} \pm \nabla \times (G^\pm \gamma_\pm \mathbf{V}_\pm) = a_\pm \frac{n^\pm}{G^\pm} (G^\pm \gamma_\pm \mathbf{V}_\pm), \quad (9)$$

aligning the Generalized vorticities along their velocity fields, and the Bernoulli conditions

$$\nabla (G^\pm \gamma_\pm \pm \varphi) = 0 \implies \quad (10)$$

$$G^+ \gamma_+ + G^- \gamma_- = \text{const}. \quad (11)$$

In the latter,  $\varphi \neq 0$  due to the asymmetry, but gravity is ignored.

The separation constants  $a_\pm$  are related to the total system energy, and the *generalized helicities*

$$h_\pm = \int (\text{curl}^{-1} \boldsymbol{\Omega}_\pm) \cdot \boldsymbol{\Omega}_\pm d\mathbf{r}. \quad (12)$$

This set, coupled with Ion fluid Beltrami Condition:

$$\mathbf{B} + \zeta \nabla \times \mathbf{V}_i = \alpha a_i n_i \mathbf{V}_i, \quad \text{where} \quad \zeta = \left[ G_0^- \frac{m^-}{m_i} \right]^{-1} \quad (13)$$

together with Ampere's law Eq.(5) defines the BB system for an e-p-i plasma – a degenerate e-p system made somewhat asymmetric by a small fraction of mobile ions ( $\alpha \ll 1$ ,  $|\mathbf{V}_i| \ll |\mathbf{V}_\pm|$ ).

Notice that there are, in fact, two asymmetry-introducing mechanisms in the e-p-i system: different effective inertias for the positively and negatively charged particles of the bulk species is one, while the small contamination from mobile ions ( $\alpha \neq 0$ ,  $\mathbf{V}_i \neq 0$ ) constitutes the other. Each one of these is responsible for creating a net "current". The structure formation mechanism explored in (Mahajan et al. 2009; Berezhiani et al. 2010) (Steinhauer & Ishida 1997), (Mahajan et al. 2001; Mahajan et al. 2002), (Ohsaki et al. 2001, 2002)), originates, for instance, in the effective inertia difference. Asymmetry between the plasma constituents increases the number of conserved helicities, and eventually translates into a

higher index Beltrami state. Later, we will explicitly show that for the degenerate e-p-i system, the Beltrami part of BB state is a *Quadruple Beltrami* state. When the second asymmetry mechanism is neglected ( $\alpha \rightarrow 0, \mathbf{V}_i \rightarrow 0$ ) a *Triple Beltrami* State follows (Bhattacharjee et al. 2003; Mahajan & Lingam 2015).

It should also be mentioned that initial asymmetry in densities ( $\alpha \neq 0$ ) can also contribute to an "effective inertia" difference in the electron-positron plasma making  $G^- \neq G^+$ ; the index of the Beltrami system, however, is contingent on the simultaneous presence of both asymmetries (see Appendix A).

#### 4 The Quadruple Beltrami system

An appropriate but tedious manipulation of the set Eq.s (5)-(13), carried out in Appendices A and B, leads us to an explicit quadruple Beltrami equation obeyed by the Ion Fluid Velocity  $\mathbf{V}_i$  (the Beltrami index is measured by the highest number of *curl* operators). Written schematically as

$$\begin{aligned} \nabla \times \nabla \times \nabla \times \nabla \times \mathbf{V}_i - b'_1 \nabla \times \nabla \times \nabla \times \mathbf{V}_i + \\ + b'_2 \nabla \times \nabla \times \mathbf{V}_i - b'_3 \nabla \times \mathbf{V}_i + b'_4 \mathbf{V}_i = 0. \end{aligned} \quad (14)$$

Equation (14) was derived in the incompressible approximation, and for  $\gamma_+ \sim \gamma_- \equiv 1$ . The  $b'$  coefficients are defined in Appendix B. Incompressibility assumption is expected to be adequate for outer layers of compact objects, though, compressibility effects can be significant e.g. in the atmospheres of pre-compact stars (Berezhiani et al. 2015a). The ion fluid velocity and the magnetic field are related to the e-p plasma average bulk fluid velocity

$$\mathbf{V} = \frac{1}{2} [(1-\alpha)\mathbf{V}_+ + \mathbf{V}_-], \quad (15)$$

through

$$\begin{aligned} \mathbf{V} = \\ = \eta (2\beta G_0^+ \nabla \times \nabla \times \mathbf{B} - [a_+(1-\alpha)\beta - a_-] \nabla \times \mathbf{B}) + \\ + \eta ([1 + (1-\alpha)\beta] \mathbf{B}) - \\ - \alpha\beta \nabla \times \mathbf{V}_i + \frac{\alpha}{2} [a_+(1-\alpha)\beta - a_-] \mathbf{V}_i \end{aligned} \quad (16)$$

with  $\eta \equiv [a_+(1-\alpha)\beta + a_-]^{-1}$  and  $\beta = G_0^-/G_0^+$ .

Once Eq.(14) is solved for  $\mathbf{V}_i$ , the vector fields  $\mathbf{B}$  and  $\mathbf{V}$  can be determined from Eqs.(13)-Eq.(16). We have

chosen to work, here, with the more familiar e-p plasma bulk velocity  $\mathbf{V}$  rather than the normalized momenta  $\mathbf{P}_\pm = G^\pm(n^\pm) \mathbf{V}_\pm$ . The e-p-i system is symmetric in  $\mathbf{B}$ ,  $\mathbf{V}$ ,  $\mathbf{V}_i$  in the sense that either of them obeys a *quadruple curl* equation. The *curl curl curl* (Triple Beltrami) equation and its solutions describing a compressible degenerate pure e-p plasma are given in Appendix C.

In Appendix B we give some illustrative examples of Quadruple [Triple] Beltrami states for degenerate e-p-i plasmas interesting for astrophysical context. Since the effect of compressibility in degenerate e-i plasma was studied in (Berezhiani et al. 2015a) we do not discuss it here and, instead, concentrate on emphasizing the effects of asymmetry stemming from the dynamic ion contamination.

The quadruple Beltrami (14) can be factorized as (details in Appendix B)

$$(curl-\mu_1)(curl-\mu_2)(curl-\mu_3)(curl-\mu_4) \mathbf{V}_i = 0, \quad (17)$$

where  $\mu_i$ -s define the coefficients in Eq.(14) and are the functions of  $\alpha$ ,  $\beta$ ,  $n_0^-$  and the degeneracy-determined mass factor  $G_0^+$ . The general solution of Eq.(17) is a sum of four Beltrami fields  $\mathbf{F}_k$  (solutions of Beltrami Equations  $\nabla \times \mathbf{F}_k = \mu_k \mathbf{F}$ ) while eigenvalues ( $\mu_k$ ) of the *curl* operator are the solutions of the fourth order equation

$$\mu^4 - b'_1 \mu^3 + b'_2 \mu^2 - b'_3 \mu + b'_4 = 0. \quad (18)$$

An examination of the various  $b'$  coefficients of (18), displayed in detail in (A10,A11) for the most relevant limit  $\alpha \ll 1$ , reveal the most interesting and important result of this enquiry. Though the inverse scales, determined by  $b'_1$ ,  $b'_2$ , and  $b'_3$ , do get somewhat modified by  $\alpha \ll 1$  corrections, it is the inverse scale associated with  $b'_4$  that is most profoundly affected; being proportional to  $\alpha$ , it tends to become small, i.e., *the corresponding scale length becomes large* as  $\alpha$  approaches zero; the scale length becomes strictly infinite for  $\alpha = 0$ , and disappears reducing (18) to a triple Beltrami system. Thus the ion contamination-induced asymmetry may lead to the formation of macroscopic structures through creating an intermediate/large length scale, much larger than the intrinsic scale skin depths, and less than the system size. The possible significance and importance of this somewhat natural mechanism (a small ion contamination is rather natural) for creating Macro-structures in astrophysical objects, could hardly be overstressed. It is important to note that this mechanism operates for all levels of degeneracy (the range of  $R_0^-$  was irrelevant).



This new macroscopic scale can be “determined” by dominant balance arguments; as the scale gets larger,  $|\nabla|$  gets smaller, and the dominant balance will be between the last terms of (18), yielding [we remind the reader, that all lengths are normalized to the  $\lambda_{\text{eff}}$ , and  $\zeta \gg 1$  even for ultra-relativistic case]

$$L_{\text{macro}} = \frac{|b'_3|}{|b'_4|} = \frac{A}{\alpha} \quad (19)$$

where

$$A = \zeta \frac{|(a_+ - a_-)[1 - \frac{\alpha}{\zeta}(G_0^+)] + \frac{\alpha}{\zeta} a_i [(G_0^+) - a_+ a_-]|}{|a_i(a_+ - a_-) - a_+ a_-|} \quad (20)$$

is a somewhat complicated function of the plasma parameters.

Assuming that the densities of the e-p-i plasmas of interest are such that  $\alpha G_0^+ / \zeta = \alpha \beta (G_0^+)^2 \frac{m_-}{m_i} \leq \alpha \ll 1$  [ $e(p)$  component density range is within  $(10^{25} - 10^{32}) \text{ cm}^{-3}$ ], we can simplify  $A$  when both  $a_+ \ll a_i$  and  $a_- \ll a_i$ .

(i) When  $a_+$  and  $a_-$  are not equal the simplified expression for dimensionless

$$L_{\text{macro}} \sim \frac{\zeta}{\alpha} \left| \frac{1}{a_i} + \frac{\alpha}{\zeta} \frac{(G_0^+) - a_+ a_-}{a_+ - a_-} \right| \quad (21)$$

for  $a_i \leq \zeta$  satisfies  $L_{\text{macro}} \gg 1$ . While

(ii) when  $a_+ \sim a_- = a \neq (G_0^+)^{1/2}$

$$L_{\text{macro}} \sim \frac{a_i}{a^2} |(G_0^+) - a^2| \gg 1 \quad (22)$$

for all  $a_i \gg a$ .

Without ion contamination ( $\alpha = 0$ ), the degenerate e-p system is still capable of creating length scales larger than the non-relativistic skin depths through the degeneracy-enhanced inertia of the light particles. Notice that even with equal effective masses ( $G^- = G^+ \simeq G(n)$  at equal electron-positron temperature), inertia change due to degeneracy can cause asymmetry in  $e(p)$  fluids [see (Mahajan & Lingam 2015)].

This, perhaps, is the right juncture to summarize the scale hierarchy encountered in this paper:

1) For a pure electron-positron plasma, the equilibrium is *triple Beltrami* with the following fundamental three scales; system size  $L$ , and the two intrinsic scales (electron and positron skin depths).

2) The e-p skin depths, microscopic in a non degenerate plasma, can become much larger due to degeneracy effects and could be classified as meso-scales,  $l_{\text{meso}}$  [see Appendix C, Eq.(C5)].

3) When a dynamic low density ion species is added, the equilibrium becomes *quadruple Beltrami* with a new additional scale,  $L_{\text{macro}}$ . Although the exact magnitude of this scale is complicated [Eq. (21)], its origin is entirely due to the ion contamination; this scale disappears as the ion concentration  $\alpha$  goes to zero. Both the larger ion mass and low density contribute towards boosting  $L_{\text{macro}}$  [see Appendix A].

4) The meso-scale  $l_{\text{meso}}$  cannot become very large but for some special constraints on the Beltrami parameters, for instance, if  $a_- \neq a_+$  and both  $a_{\pm} \ll 1$  or the condition (C6) is satisfied.

## 5 Conclusions and Summary

In the present paper we derived *Quadruple* [*Triple*] Beltrami relaxed states in e-p-i plasma with classical ions, and degenerate electrons and positrons. Such a mix is often met in both astrophysical and laboratory conditions.

The presence of the mobile ion component has a striking qualitative effect; it converts, what would have been, a *triple* Beltrami state to a new *quadruple* Beltrami state. In the process, it adds structures at a brand new macroscopic scale  $L_{\text{macro}}$  (absent when ion concentration is zero) that is much larger than the intrinsic skin depth ( $\lambda = c \sqrt{\frac{m_-}{8\pi n_0 e^2}}$ ) of the lighter components.

Though primarily controlled by the mobile ion concentration,  $L_{\text{macro}}$  also takes cognizance of the electron and positron inertias that could be considerably enhanced by degeneracy. In fact even in the absence of ions ( $L_{\text{macro}} \rightarrow \text{infinity}$ ), the Beltrami states could be characterized by what could be called meso-scales – the temperature and degeneracy-boosted effective skin depths  $\lambda_{\text{eff}}^{\pm}$  larger than  $\lambda$  [according to (6)  $\lambda_{\text{eff}}^{\pm} / \lambda = \sqrt{G_0^{\pm}} > 1$  and  $1 < \sqrt{G_0^{\pm}} < 5.6$  for densities  $(10^{25} - 10^{32}) \text{ cm}^{-3}$ ]. At the same time it has to be emphasized here that for larger scale to exist we do need an entirely different mechanism – a dynamic ion-species with a much lower density and higher rest mass (justified by observations for many astrophysical objects plasmas) – this scale corresponds to the ion skin depth enhanced, dramatically, by low density [ $\lambda_i = (\alpha m_- / m_i)^{-1/2} \lambda \gg \lambda$ ].

The creation of these new intermediate scales (between the system size, and  $\lambda$ ) adds immensely to the richness of the structures that such an e-p-i plasma can sustain; many more pathways become accessible for energy transformations. Such pathways could help us better understand a host of quiescent as well as explosive astrophysical phenomena - eruptions, fast/transient

outflow and jet formation, magnetic field generation, structure formation, heating etc. At the same time, results found in present manuscript indicate that when the star contracts, for example, its outer layers keep the multi-structure character although density in the structures, as shown in (Berezhiani et al. 2015a), becomes defined by lighter components degeneracy pressure. Future studies will include a detailed investigation of present model to explore the evolution of multi-structure stellar outer layers while contracting, cooling.

## 6 Acknowledgements

Authors acknowledge special debt to the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. The work of S.M.M. was supported by USDOE Contract No. DEFG 03-96ER-54366.

## References

- Akbari-Moghanjoughi, M. 2013 *Phys. Plasmas* **20**, 042706.
- Aksenov, A.G., Ruffini, R. and Vereshchagin, G.V. 2010 *Phys. Rev. E* **81** 046401.
- Begelman, M.C., Blandford, R.D., and Rees, M.D. *Rev. Mod. Phys.* **56** 255 (1984).
- Beloborodov, A.M. and Thompson, C. *Astrophys. J.* **657**, 967 (2007).
- Berezhiani, V.I. and Mahajan, S.M. *Phys. Rev. Lett.* **73**, 1110 (1994); *Phys. Rev. E* **52** 1968 (1995).
- Berezhiani, V.I., El-Ashry, M.I. and Mofiz, U.A. *Phys. Rev. E* **50**, 448 (1994).
- Berezhiani, V.I., Mahajan, S.M., Yoshida, Z. and Ohashi, M. 2002 *Phys. Rev. E* **65** 047402; Berezhiani, V.I., Mahajan, S.M. and Yoshida, Z. *Phys. Rev. E* **78** 066403 (2008).
- Berezhiani, V.I., Mahajan, S.M. and Shatashvili, N.L. *Phys. Rev. A* **81**, 053812 (2010); *ibid. J. of Plasma Phys.* **76**, 467 (2010).
- Berezhiani, V.I., Shatashvili, N.L. and Mahajan, S.M. *Phys. Plasmas* **22**, 022902 (2015a).
- Berezhiani, V.I., Shatashvili, N.L. and Tsintsadze, N.I. *Physica Scripta* **90**(6), 068005 (2015b).
- Bhattacharjee, A. and Dewar, R.L. *Phys. Fluids* **25**, 887 (1982).
- Bhattacharyya, R., Janaki, M.S., Dasgupta, B. *Phys. Lett. A* **315**, 120 (2003).
- Cercignani, C. and Kremer, G.M. 2002 *The relativistic Boltzmann equation: theory and applications* Birkhäuser, Basel; chapter 3.
- Chandrasekhar, S. *Astrophys. J.* **74**, 81 (1931); *Mon. Not. R. Astron. Soc.* **95**, 207 (1935).
- Chandrasekhar, S. *An Introduction to the Study of Stellar structures*, Chicago (Dover Publications, 1939).
- Dennis, G.R., Hudson, S.R., Dewar, R.L. and Hole, M.J. *Phys. Plasmas* **21**, 042501 (2014).
- Dunne, M. A high-power laser fusion facility for Europe *Nature Phys.* **2** 2 (2006).
- Gedalin, M., Melrose, D.B. and Gruman, E. *Phys. Rev. E* **57** 3399 (1998).
- Gedalin, M. *Phys. Rev. Lett.* **76**, 3340 (1996).
- Halder, M.M., Akter, S., Duha, S.S. and Mamun, A.A. *Cent. Eur. J. Phys.* **10**(5), 1168 (2012).
- Han, W.B., Ruffini, R. and Xue, S.S. *Phys. Rev. D* **86** 084004 (2012).
- Hazeltine, R.D. and Mahajan, S.M. *Astrophys. J.* **567**, 1262 (2002).
- Holcomb, K.A. and Tajima, T. *Phys. Rev. D* **40** 3809 (1989).
- Iqbal, N., Berezhiani, V.I. and Yoshida, Z. *Phys. Plasmas* **15**, 032905 (2008).
- Khan, S.A. *Phys. Plasmas* **19**, 014506 (2012).
- Koester, D. and Chanmugam, G. *Rep. Prog. Phys.* **53**, 837 (1990).
- Lakhina, G.S. and Buti, B. *Astrophys. Space Sci.* **79**, 25 (1991).
- Lominadze, J.G., Machabeli, G.Z., Melikidze, G.I. and Pataraya, A.d. *Fiz. Plazmy* **12**, 1233 (1986) [*Sov. J. Plasma Phys.* **12**, 712 (1986)].
- Landau, L.D. and Lifshitz, E.M. 1980 *Statistical Physics* Pergamon Press Ltd.
- Mahmood, S. Safeer Sadiq, and Q. Haque. *Phys. Plasmas* **20**, 122305 (2013).
- Michel, F.C. *Theory of Neutron Star Magnetospheres*, University of Chicago Press, Chicago, (1991).
- Michel, F.C. *Rev. Mod. Phys.* **54**, 1 (1982).
- Misra, A.P. and Shukla, P.K. *Phys. Rev. E* **85**, 026409 (2012).
- Mahajan, S.M. and Yoshida, Z. *Phys. Rev. Lett.* **81**, 4863 (1998).
- Mahajan, S.M., Miklaszewski, R., Nikolskaya, K.I. and Shatashvili, N.L. *Phys. Plasmas* **8**, 1340 (2001).
- Mahajan, S.M., Nikolskaya, K. I., Shatashvili, N.L. and Yoshida, Z. *Astrophys. J.* **576**, L161 (2002).
- Mahajan, S.M. *Phys. Rev. Lett.* **90**, 035001 (2003).
- Mahajan, S.M., Shatashvili, N.L., Mikeladze, S.V. and Sigua, K.I. *Astrophys. J.* **634**, 419 (2005); *Phys. Plasmas* **13**, 062902 (2006).
- Mahajan, S.M., Shatashvili, N.L. and Berezhiani, V.I. *Phys. Rev. E* **80**, 066404 (2009).
- Mahajan, S.M. and Krishan, V. *Mon. Not. R. Astron. Soc.* **359**, L29 (2005).
- Mahajan, S.M. and Lingam, M. *Phys. Plasmas*, **22**(9), 092123 (2015).
- Mikaberidze, G. and Berezhiani, V.I. *Phys. Lett.* **379**, **42**, 2730 (2015).
- Mourou, G.A., Tajima, T. and Bulanov, S. V. *Rev. Mod. Phys.* **78** 309 (2006).
- Oliveira, S.R. and Tajima, T. *Phys. Rev. E* **52**, 287 (1995).
- Ohsaki, S., Shatashvili, N.L., Yoshida, Z. and Mahajan, S.M. *Astrophys. J.* **559**, L61 (2001); *Astrophys. J.* **570**, 395 (2002).
- Pino, J., Li, H. and Mahajan, S.M. *Phys. Plasmas* **17**, 112112 (2010).
- Rizzato, F.B. *J. Plasma Phys.* **40**, 289 (1988).
- Ruderman, M.A. and Sutherland, P.G. *Astrophys. J.* **196**, 51 (1975).
- Russo, G. *Astrophys. J.* **334** 707 (1988).

- 
- Shapiro, L. and Teukolsky, S.A. *Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects*, (John Wiley and Sons, New York, 1973).
- Shukla, P.K. and Mahajan, S.M. Phys. Lett. A, **328**, 185 (2004a); *ibid.* Phys. Scr. **T113**, 151 (2004b).
- Shukla, P.K. and Eliasson, B. Phys. Usp. **53**, 51 (2010).
- Shukla, P.K. and Eliasson, B. Rev. Mod. Phys. **83**, 885 (2011).
- Steinhauer, L.C. and Ishida, A. Phys. Rev. Lett. **79**, 3423 (1997).
- Sturrock, P.A. Astrophys. J. **164**, 529 (1971).
- Sudan, R.N. Phys. Rev. Lett. **42**, 1277 (1979).
- Tajima, T. Eur. Phys. J. Sp. Top. 223(6), 1037 (2014).
- Taylor, J.B. Phys. Rev. Lett. **33**, 1139 (1974); Rev. Mod. Phys. **58**, 741 (1986).
- Tsintsadze, N.L. , Shukla, P.K. and Stenflo, L. Eur. Phys. J. D. **23** 109 (2003).
- Uddin, M.J., Alam, M.S. and Mamun, A.A. Phys. Plasmas **22**, 022111 (2015).
- Weinberg, S. 1972 *Gravitation and Cosmology* (Weley, New York)
- Wang, Y., Shukla, P.K. and Eliasson, B. Phys. Plasmas **20** 013103 (2013).
- Woltjer, L. Proc. Natl. Acad. Sci. U. S. **44**, 489 (1958).
- Yanovsky, V., Chvykov, V., Kalinchenko, G., Rousseau, P., Planchon, T., Matsuoka, T., Maksimchuk, A., Nees, J., Cheriaux, G. , Mourou G and Krushelnick K Opt. Exp. **16** 2109 (2008).
- Yoshida, Z. and Mahajan, S.M. J. Math. Phys. **40**, 5080 (1999).
- Yoshida, Z. Comm. Nonlinear Sci. Num. Sim. **17**, 2223 (2011).



## A Appendix - Derivation of Quadruple Beltrami Equation

The Ampere's law generally can be written in dimensionless variables as:

$$\nabla \times \mathbf{B} = \frac{1}{2} \left[ \alpha \frac{N_i}{N_{0i}} \mathbf{V}_i + (1 - \alpha) \frac{N^+}{N_0^+} \mathbf{V}_+ - \frac{N^-}{N_0^-} \mathbf{V}_- \right] \quad (\text{A1})$$

and

(i) if  $\alpha = 1$  (e-i plasma, quasineutralisty reads as  $N_i = N^- = N$ ) we have:

$$\mathbf{V}_- \equiv \mathbf{V}_e = \mathbf{V}_i - \frac{2}{N} \nabla \times \mathbf{B}$$

leading to Double Beltrami (DB) states in e-i plasma with degenerate electrons (Berezhiani et al. 2015a);

(ii) while when  $\alpha = 0$  (purely symmetric e-p plasma, quasineutralisty reads as  $N^+ = N^- = N$ ) we have:

$$\mathbf{V}_- - \mathbf{V}_+ = -\frac{2}{N} \nabla \times \mathbf{B}$$

that shall lead to higher Beltrami states when inertia effects in electron and positron fluids are taken into account [similar effect was discussed for relativistic non-degenerate plasmas in (Yoshida & Mahajan 1999; Iqbal et al. 2008; Bhattacharjee et al. 2003; Mahajan & Lingam 2015)].

Observations show that ion fluid fraction can be small ( $\alpha \ll 1$ ); also ion fluid velocity is much smaller than those for lighter elements – electron and positron fluids [ $\mathbf{V}_i \ll \mathbf{V}_-, \mathbf{V}_+$ ] and, hence, one can imagine that the mobility of ions can be ignored in most of the cases (Oliveira & Tajima 1995) except when  $\alpha = 1$  where, as it was shown in (Mahajan et al. 2001; Mahajan et al. 2002; Mahajan et al. 2005, 2006), flow effects can be crucial in creating the structural richness in astrophysical environments as well as in the heating/cooling processes, Generalized Dynamo theory and flow acceleration phenomena. The case of  $\alpha = 1$  - pure e-i plasma with degenerate electrons was already studied in (Berezhiani et al. 2015a) and it was shown that when ignoring inertia effects in electron fluid the Double Beltrami states are accessible in the system. Hence, it is expected, that when ion fluid velocity is not neglected in e-p-i plasma with degenerate electrons and positrons number of relaxed states can be either 2 (when ignoring degenerate  $e(p)$  fluids inertia effects although  $G^- \neq G^+$ ) or 4 (when degenerate fluids inertia effects are taken into account); at the same time neglecting the ion flow effects ( $\alpha \rightarrow 0, \mathbf{V}_i \rightarrow 0$ ) we shall obtain the Single Beltrami state in former situation and the Triple Beltrami States in latter case – this problem is a scope of our study below [see (Mahajan & Lingam 2015) and its results].

Let us now show how Beltrami states may acquire new structures due to degeneracy or/and the small fraction of mobile ions. We will study an incompressible e-p-i plasma with the simplifying assumption  $\gamma_+ \sim \gamma_- \equiv 1$  that reduces the Generalized Bernoulli Conditions to  $G^+ + G^- = \text{const}$ . The Ampere's law (A1), in dimensionless variables, is written as

$$\nabla \times \mathbf{B} = \frac{1}{2} [\alpha \mathbf{V}_i + (1 - \alpha) \mathbf{V}_+ - \mathbf{V}_-] . \quad (\text{A2})$$

In terms of the e-p plasma bulk flow average velocity

$$\mathbf{V} = \frac{1}{2} [(1 - \alpha) \mathbf{V}_+ + \mathbf{V}_-] \quad (\text{A3})$$

one can express the *Generalized Momenta* for positron and electron fluids as  $[\mathbf{P}_\pm = G_0^\pm(n_0^\pm) \mathbf{V}_\pm]$  :

$$\mathbf{P}_+ = \frac{G_0^+}{1 - \alpha} (\mathbf{V} + \nabla \times \mathbf{B} - \frac{\alpha}{2} \mathbf{V}_i) ; \quad \mathbf{P}_- = G_0^- (\mathbf{V} - \nabla \times \mathbf{B} + \frac{\alpha}{2} \mathbf{V}_i) , \quad (\text{A4})$$

Introducing  $\beta \equiv \frac{G_0^-}{G_0^+}$  and using Eqs. (A4) in Eqs. (9), straightforward algebra leads to:

$$\begin{aligned} \mathbf{V} = \eta \left( 2\beta G_0^+ \nabla \times \nabla \times \mathbf{B} - [a_+(1-\alpha)\beta - a_-] \nabla \times \mathbf{B} + [1 + (1-\alpha)\beta] \mathbf{B} \right) - \\ - \alpha\beta \nabla \times \mathbf{V}_i + \frac{\alpha}{2} [a_+(1-\alpha)\beta - a_-] \mathbf{V}_i \end{aligned} \quad (\text{A5})$$

with  $\eta \equiv [a_+(1-\alpha)\beta + a_-]^{-1}$ .

We have to add the Ion flow Beltrami condition (13) written for incompressible case as

$$\mathbf{B} + \zeta \nabla \times \mathbf{V}_i = \alpha a_i \mathbf{V}_i \quad (\text{A6})$$

to close the system of equations for incompressible e-p-i degenerate plasma.

Plugging the Eq. (A5) into the Eq.-s (A4) and then using them in Eq.-s (9) we get

$$\begin{aligned} 2\beta (G_0^+)^2 \nabla \times \nabla \times \nabla \times \mathbf{B} - 2G_0^+ \alpha_1 \nabla \times \nabla \times \mathbf{B} + \alpha_2 \nabla \times \mathbf{B} - \alpha_3 \mathbf{B} = \\ = \alpha (G_0^+)^2 \nabla \times \nabla \times \mathbf{V}_i - \alpha (G_0^+) \alpha_1 \nabla \times \mathbf{V}_i - \alpha (1-\alpha) a_+ a_- \mathbf{V}_i, \end{aligned} \quad (\text{A7})$$

where

$$\alpha_1 = [a_+(1-\alpha)\beta - a_-], \quad \alpha_2 = [1 + (1-\alpha)\beta] G_0^+ - 2\beta(1-\alpha) a_+ a_-, \quad \alpha_3 = (1-\alpha) (a_+ + a_-). \quad (\text{A8})$$

The equation (A7) for immobile ions ( $\mathbf{V}_i \equiv 0$ ) will eventually give the so called "Triple Beltrami" equation for the magnetic field  $\mathbf{B}$  (i.e. l.h.s. of Eq.(A7)  $\equiv 0$ ).

Let us now simplify the equations when the ion density is just a small fraction of the density of the light species, i.e.  $\alpha \ll 1 \implies (1-\alpha) \rightarrow 1$ ;  $[1 + (1-\alpha)\beta] \rightarrow 2$ . After tedious but simple algebra, one obtains, in this limit, the quadruple Beltrami equation for  $\mathbf{V}_i$ :

$$(G_0^+)^2 \nabla \times \nabla \times \nabla \times \nabla \times \mathbf{V}_i - b_1 (G_0^+) \nabla \times \nabla \times \nabla \times \mathbf{V}_i + b_2 (G_0^+) \nabla \times \nabla \times \mathbf{V}_i - b_3 \nabla \times \mathbf{V}_i + b_4 \mathbf{V}_i = 0, \quad (\text{A9})$$

where

$$b_1 = (a_+ - a_-) + \frac{1}{2} a_i (G_0^+); \quad b_2 = [(G_0^+) - a_+ a_-] + \frac{\alpha}{\zeta} a_i (a_+ - a_-) + \frac{\alpha}{\zeta} (G_0^+); \quad (\text{A10})$$

$$b_3 = (a_+ - a_-) [1 - \frac{\alpha}{\zeta} (G_0^+)] + \frac{\alpha}{\zeta} a_i [(G_0^+) - a_+ a_-]; \quad b_4 = \frac{\alpha}{\zeta} [a_i (a_+ - a_-) - a_+ a_-]. \quad (\text{A11})$$

Notice, that in above relations the terms with coefficient  $(\alpha/\zeta)$  will vanish for either  $\alpha \rightarrow 0$  (when there is no fraction of ions at all!) or  $m_i \rightarrow \infty$  (which means that ions are immobile!). In such case we arrive to Triple Beltrami Equations for  $\mathbf{B}$  and  $\mathbf{V}$  as mentioned above. Also, it is interesting to note that due to mobile ions (i.e. when  $(\alpha/\zeta) \neq 0$ ) the large scale is automatically there due to  $b_4 \neq 0$  in Eq.(A9). We will show this in detail in the Appendix B.

Solving the Eq.(A9) for  $\mathbf{V}_i$  and plugging it into (A6) we will get the equation for  $\mathbf{B}$ ; for the pure incompressible e-p plasma it is better to use Eq.(A5) directly to find the magnetic field  $\mathbf{B}$ .

## B Appendix - Scale separation in degenerate e-p-i Plasma – Quadruple Beltrami Structures

Here we give some illustrative examples of Quadruple [Triple] Beltrami states for degenerate e-p-i [e-p] plasmas interesting for astrophysical context.

Introducing

$$b'_1 = \frac{b_1}{G_0^+}; \quad b'_2 = \frac{b_2}{G_0^+}; \quad b'_3 = \frac{b_3}{(G_0^+)^2}; \quad b'_4 = \frac{b_4}{(G_0^+)^2} \quad (\text{B1})$$

the Eq. (A9) reads as:

$$\nabla \times \nabla \times \nabla \times \nabla \times \mathbf{V}_i - b'_1 \nabla \times \nabla \times \nabla \times \mathbf{V}_i + b'_2 \nabla \times \nabla \times \mathbf{V}_i - b'_3 \nabla \times \mathbf{V}_i + b'_4 \mathbf{V}_i = 0, \quad (\text{B2})$$

Equation (B2) can be written as

$$(curl - \mu_1)(curl - \mu_2)(curl - \mu_3)(curl - \mu_4) \mathbf{V}_i = 0, \quad (\text{B3})$$

where  $\mu_k$ -s define the coefficients in Eq. (B2) as

$$\begin{aligned} b'_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4, & b'_2 &= \mu_1 \mu_2 + \mu_1 \mu_3 + \mu_1 \mu_4 + \mu_2 \mu_3 + \mu_2 \mu_4 + \mu_3 \mu_4, \\ b'_3 &= \mu_1 \mu_2 \mu_3 + \mu_1 \mu_2 \mu_4 + \mu_1 \mu_3 \mu_4 + \mu_2 \mu_3 \mu_4, & b'_4 &= \mu_1 \mu_2 \mu_3 \mu_4. \end{aligned} \quad (\text{B4})$$

The general solution of Eq.(B3) is a sum of four Beltrami fields  $\mathbf{F}_k$  (solutions of Beltrami Equations  $\nabla \times \mathbf{F}_k = \mu_k \mathbf{F}$ ) while eigenvalues ( $\mu_k$ ) of the *curl* operator are the solutions of the fourth order equation

$$\mu^4 - b'_1 \mu^3 + b'_2 \mu^2 - b'_3 \mu + b'_4 = 0. \quad (\text{B5})$$

Since  $b'_4 \rightarrow 0$  for our case of study – e-p-i plasma with small fraction of mobile ions:  $\alpha \ll 1$ ;  $m^- \ll m_i \implies$  that the Eq.(B5) can be reduced to

$$\mu (\mu - \mu_2) (\mu - \mu_3) (\mu - \mu_4) = 0. \quad (\text{B6})$$

solving of which gives  $\mu_1 \simeq 0$  – hence, the large scale structure existence is automatically guaranteed in such plasma due to the presence of small fraction of mobile ions (asymmetry due to small ion contamination); we stress here that this scale is always there.

Now we will have to solve the remaining equation:

$$\mu^3 - b'_1 \mu^2 + b'_2 \mu - b'_3 = 0. \quad (\text{B7})$$

There are 3 possible simple, analytically tractable, scenarios:

(i) If

$$(a_+ - a_-)[1 - \frac{\alpha}{\zeta} (G_0^+)] = -\frac{\alpha}{\zeta} a_i [(G_0^+) - a_+ a_-] \quad (\text{B8})$$

then  $b'_3 \simeq 0$ , and eq. (B7) reduces to:

$$\mu (\mu + a_1) (\mu - a_2) = 0, \quad (\text{B9})$$

solving of which we find that  $\mu_2 \simeq 0$  [another large large scale now due to the asymmetry in degeneracy induced inertias of electrons and positrons] is also guaranteed in such plasma and two other short-scales are defined by Beltrami Parameters  $a_+$  and  $a_-$  as:  $\mu_3 = -a_1$ ;  $\mu_4 = a_2$ , where

$$a_1 = \frac{1}{2}b'_1 \mp \frac{1}{2}\sqrt{(b'_1)^2 - 4b'_2} \quad \text{and} \quad a_2 = \frac{1}{2}b'_1 \pm \frac{1}{2}\sqrt{(b'_1)^2 - 4b'_2}. \quad (\text{B10})$$

In pure e-p plasma above conditions reduce to  $a_+ \sim a_-$  ( $a_1 = a_+$  while  $a_2 = a_-$ ) and eventually there are only 3 scales in total (defining equation is *Triple Beltrami*).

(ii) If  $\alpha$  and  $\beta$  and other parameters are such that both  $b'_1 \simeq 0$  and  $b'_3 \simeq 0$ , then  $b'_2 = \mu_3 \mu_4 \neq 0$ . Here again one of the roots is zero (let it be  $\mu_2 = 0$ ), while other two satisfy the relations:

$$\mu_3 + \mu_4 = 0; \quad \mu_3 \mu_4 = b'_2. \quad (\text{B11})$$

Hence, also in this case length-scales are vastly separated (two scales are of similar range (short scales) and one is significantly large-scale):  $|\mu_2| = |\mu_3|$ ;  $\mu_2 \simeq 0$ . For such scenario Beltrami coefficients  $a_{\pm}$  must be such that  $b'_2 < 0$  is maintained; for pure e-p plasma this is the case when  $a_+ = a_- = a > (G/n)^{1/2}$ .

(iii) If  $\alpha$  and  $\beta$  and other parameters are such that all  $b'_1 \simeq 0$ ,  $b'_2 \simeq 0$  and  $b'_3 \simeq 0$  can be established in addition to guaranteed  $b'_4 \simeq 0$ , then all the scale parameters  $\mu_1, \mu_2, \mu_3, \mu_4$  become close to zero – no separation of length scales. For pure e-p plasma this is the case when  $a_+ = a_- = a = (G/n)^{1/2}$ .

Thus, we can conclude, that for a rather big range of parameters there is a guaranteed scale separation in e-p-i plasma with degenerate lighter components.

### C Appendix - Scale separation in compressible degenerate e-p Plasma – Triple Beltrami Structures

Note that with no fraction of ions [ $\alpha = 0$ ,  $\beta = 1$ ] there is no charge separation and the scalar potential  $\varphi \equiv 0$  in Eq.-s (3 - 4); hence,  $n^- = n^+ = n$ ,  $\mathbf{B} = \nabla \times \mathbf{A}$ ;  $\mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}$  and the assumption of  $T_{\pm} \rightarrow 0$  leads to the initial temperature symmetry between species; we have consequently  $G^- = G^+ = G(n)$ . The existence of soliton-like electromagnetic (EM) distributions in such a fully degenerate electron-positron plasma was studied in (Berezhiani et al. 2015b).

Then, for pure compressible e-p plasma, if  $\nabla[G^{\pm}(n^{\pm})]$  is at a much slower rate than the spatial derivatives of  $\mathbf{B}$  and  $\mathbf{V}_{\pm}$ , we can write instead of (A5) following relation [with corresponding  $\eta = 1/(a_+ + a_-)$ ]:

$$\mathbf{V} = \eta \left( 2 \frac{G}{n} \nabla \times \frac{1}{n} \nabla \times \mathbf{B} - \frac{a_+ - a_-}{n} \nabla \times \mathbf{B} \right) + \eta \frac{2\mathbf{B}}{n} \quad (\text{C1})$$

and instead of (A7) we obtain:

$$\nabla \times \left( \frac{G}{n} \right) \nabla \times \left( \frac{1}{n} \right) \nabla \times \mathbf{B} - \kappa_1 \nabla \times \left( \frac{1}{n} \right) \nabla \times \mathbf{B} + \kappa_2 \nabla \times \left( \frac{G}{n} - a_+ a_- \right) \mathbf{B} - \kappa_3 \mathbf{B} = 0, \quad (\text{C2})$$

where

$$\kappa_1 = (a_+ - a_-); \quad \kappa_2 = \frac{1}{G}; \quad \kappa_3 = \frac{a_+ - a_-}{2G}. \quad (\text{C3})$$

After simple algebra we arrive to the defining equation for  $\mathbf{V} = \frac{1}{2}(\mathbf{V}_+ + \mathbf{V}_-)$ :

$$\left( \frac{G}{n} \right) \nabla \times \left( \frac{1}{n} \right) \nabla \times \nabla \times \mathbf{V} - \kappa_1 \left( \frac{1}{n} \right) \nabla \times \nabla \times \mathbf{V} + \kappa_2 \nabla \times \left( \frac{G}{n} - a_+ a_- \right) \left( \frac{n}{G} \right) \mathbf{V} - \kappa_3 \mathbf{V} = 0. \quad (\text{C4})$$

Solution of Eq.-s (C2) and (C4) is possible following the scenarios given in Appendix B, solutions will be similar to those given after Eq.(B7), just density dependent. Estimation for the large scale  $l_{\text{meso}}$  in case of pure degenerate e-p plasma, derived from the dominant balance, gives:

$$l_{\text{meso}} = \frac{|\kappa_2|}{|\kappa_3|} |(G/n) - a_+ a_-| = 2 \frac{|(G/n) - a_+ a_-|}{|a_+ - a_-|} \gg 1 \quad (\text{C5})$$

if

$$a_+ = a_- = a \neq \left( \frac{G(n)}{n} \right)^{1/2}. \quad (\text{C6})$$

Hence, whenever the local density satisfies this condition there is a guaranteed scale separation in the degenerate e-p plasma with at least one large scale present.